

The Theory of Orthogonal R -Separation for Helmholtz Equations

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We develop the theory of orthogonal R -separation for the Helmholtz equation on a pseudo-Riemannian manifold and show that it, and not ordinary variable separation, is the natural analogy of additive separation for the Hamiltonian–Jacobi equation. We provide a coordinate-free characterization of R -separation in terms of commuting symmetry operators.

1. INTRODUCTION

Let $\{y^j\}$ be a local coordinate system on the pseudo-Riemannian manifold V_n . The *Helmholtz equation* in these coordinates is

$$\Delta\psi(\mathbf{y}) = E\psi(\mathbf{y}) \quad (1.1)$$

where E is a nonzero constant and Δ is the Hamiltonian or Laplace–Beltrami operator [1]

$$\Delta = \frac{1}{\sqrt{g}} \sum_{i,j=1}^n \partial_i (\sqrt{g} g^{ij} \partial_j). \quad (1.2)$$

Here, $\partial_j = \partial_{y^j}$, the metric on V_n is

$$ds^2 = \sum g_{ij} dy^i dy^j, \quad g = \det(g_{ij})$$

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and $\sum_k g^{ik} g_{kj} = \delta_j^i$. Closely associated with (1.1) is the *Hamilton–Jacobi equation* [2]

$$H(\partial_i W) \equiv \sum_{i,j=1}^n g^{ij} \partial_i W \partial_j W = E \quad (1.3)$$

where H is the Hamiltonian function

$$H(p_i) = \sum_{i,j=1}^n g^{ij} p_i p_j. \quad (1.4)$$

Both Δ and H are defined independent of coordinates.

It is well known that there is a close association between additively separable solutions of (1.3) in an appropriate orthogonal coordinate system and multiplicatively separable solutions of (1.1). Indeed the famous Robertson–Eisenhart condition [2], $R_{ij} = 0$, $i \neq j$, is just the requirement that a separable system for (1.3) also separate (1.1). In this paper we will develop a theory of orthogonal R -separation for the Helmholtz equation, that is, separation up to a fixed factor:

$$\psi(\mathbf{y}) = R(\mathbf{y}) \prod_{j=1}^n \psi^{(j)}(y^j). \quad (1.5)$$

(Ordinary separation corresponds to $R \equiv 1$ and trivial R -separation to $\partial_{ij} \log R = 0$ for $i \neq j$.) We will show that, despite the elegance of the Robertson condition, it is R -separation rather than the more restricted ordinary separation that is the proper analog of separation for the Hamilton–Jacobi equation. Also we will demonstrate that nontrivial R -separable systems are abundant. The extension of our methods to Helmholtz (or Schrödinger) equations with potentials is straightforward.

In Section 2 we give a precise operational definition of orthogonal R -separation and review the theory of ordinary separation for both the Helmholtz and Hamilton–Jacobi equations. In Section 3 we introduce the elementary concepts of self-adjoint and reduced self-adjoint form for second-order differential operators that commute with Δ . Section 4 is devoted to the proof of our basic result, an intrinsic characterization of R -separable coordinates for the Helmholtz equation which is in exact analogy to our previous result for the Hamilton–Jacobi equation [3]. Briefly, we show that necessary and sufficient conditions for orthogonal R -separation of (1.1) are the existence of an $n - 1$ -dimensional family of commuting symmetry operators for Δ which are in self-adjoint form and can be simultaneously diagonalized. The coordinates can be computed from the simultaneous eigenforms. In Section 5 we provide some examples of nontrivial R -separation. Section 6 is devoted to a short discussion of the significance of our results.

All our considerations are local rather than global, although Theorem 3 clearly has global implications. Any function occurring in this paper is assumed to be analytic.

2. STÄCKEL FORM AND R -SEPARATION

Let $\{x^j\}$ be an orthogonal coordinate system on the (local) pseudo-Riemannian manifold V_n . Then the metric in these coordinates takes the form

$$ds^2 = \sum_{i=1}^n H_i^2 (dx^i)^2 \quad (2.1)$$

and the Helmholtz equation becomes

$$\Delta\psi \equiv \frac{1}{h} \sum_i \partial_i (h H_i^{-2} \partial_i \psi) = E\psi \quad (2.2)$$

where $h = H_1 \cdots H_n$. In order to explain the problem posed in Section 1 and our method of solution we give here the construction to obtain R -separable solutions

$$\psi(\mathbf{x}) = R(\mathbf{x}) \prod_{j=1}^n \psi^{(j)}(x^j) \quad (2.3)$$

for (2.2) and derive conditions on the success of that construction. Let $(S_{ij}(x^i))$ be a *Stäckel matrix*, i.e., an $n \times n$ nonsingular matrix whose i th row depends only on the variable x^i , and set $S = \det(S_{ij})$. Further let $\lambda_1 = -E$, $\lambda_2, \dots, \lambda_n$ be complex parameters and define differential operators K_j , $j = 1, \dots, n$, by

$$K_j = \partial_{jj} + l_j \partial_j + m_j + \sum_{i=1}^n \lambda_i S_{ji}(x^j) \quad (2.4)$$

where l_j , m_j are functions of x^j alone and $\partial_j = \partial_{x^j}$. We say that the orthogonal coordinates $\{x^i\}$ are R -separable for the Helmholtz equation (2.2) provided there exist functions $g_i(\mathbf{x})$ and $R(\mathbf{x})$ ($R \neq 0$) such that

$$R^{-1} \Delta R - E \equiv \sum_{j=1}^n g_j(\mathbf{x}) K_j. \quad (2.5)$$

(Explicitly,

$$R^{-1} \Delta R = \frac{1}{h} \sum_i \partial_i (h H_i^{-2} \partial_i) + 2 \sum_i H_i^{-2} (\partial_i \log R) \partial_i + R^{-1} (\Delta R) \quad (2.6)$$

as an operator.) If the coordinates are R -separable then the function ψ , (2.3), is a solution of $\Delta\psi = E\psi$ whenever the $\psi^{(j)}$ satisfy the (ordinary differential) *separation equations*

$$K_j \psi^{(j)} = 0, \quad j = 1, \dots, n. \quad (2.7)$$

It follows easily from (2.4), (2.5) that a necessary condition for R -separation is

$$g_j(\mathbf{x}) = S^{j1}/S \quad (2.8)$$

where S^{j1} is the $(j, 1)$ minor of (S_{ij}) ; hence from (2.5), (2.6) the metric must be in *Stäckel form*

$$H_j^{-2} = S^{j1}/S, \quad j = 1, \dots, n. \quad (2.9)$$

It is well known that the orthogonal coordinates $\{x^i\}$ permit (additive) separation of the Hamilton–Jacobi equation

$$\sum_{j=1}^n H_j^{-2} (\partial_j W)^2 = E \quad (2.10)$$

that is, separation in the form $W = \sum_{j=1}^n W^{(j)}(x^j)$, if and only if condition (2.9) holds for some Stäckel matrix $(S_{ij}(x^i))$ [2].

However, Stäckel form is not sufficient for (product) R -separation of the Helmholtz equation. In addition we must require equality of the coefficients of ∂_j and the zeroth-order terms on each side of (2.5):

$$f_j + 2\partial_j \log R = l_j(x^j) \quad (2.11)$$

$$R^{-1}(\Delta R) = \sum H_i^{-2} m_i(x^i). \quad (2.12)$$

Here

$$f_j \equiv \partial_j f \equiv \partial_j \log(h/S). \quad (2.13)$$

Solving for R from (2.11) and substituting this expression into (2.12) we find that the separation conditions become

$$\sum_{i=1}^n H_i^{-2} (f_{ii} + \frac{1}{2} f_i^2) = \sum_i H_i^{-2} \tilde{m}_i(x^i) \quad (2.14)$$

where the \tilde{m}_i are functions of x^i alone. (Indeed, $\tilde{m}_i = -2m_i + \partial_i l_i + \frac{1}{2} l_i^2$.)

To express these conditions more simply we recall some results from Ref. [4]. Given a metric $ds^2 = \sum H_i^{-2} (dx^i)^2$ in Stäckel form, we say that the function $Q(\mathbf{x})$ is a *Stäckel multiplier* (for ds^2) if the metric $d\tilde{s}^2 = Q ds^2$ is

also in Stäckel form with respect to the coordinates $\{x^j\}$. It can be shown that Q is a Stäckel multiplier if and only if there exist functions $\psi_j = \psi_j(x^j)$ such that

$$Q(\mathbf{x}) = \sum_{j=1}^n \psi_j(x^j) H_j^{-2}. \quad (2.15)$$

Furthermore, necessary and sufficient conditions that Q be a Stäckel multiplier are

$$\partial_{jk} Q - \partial_j Q \partial_k \text{Log } H_j^{-2} - \partial_k Q \partial_j \text{Log } H_k^{-2} = 0, \quad j \neq k. \quad (2.16)$$

(Recall that necessary and sufficient conditions that ds^2 be in Stäckel form are [2]:

$$\begin{aligned} \partial_{jk} \text{Log } H_i^{-2} + \partial_j \text{Log } H_i^{-2} \partial_k \text{Log } H_i^{-2} - \partial_j \text{Log } H_i^{-2} \partial_k \text{Log } H_j^{-2} \\ - \partial_k \text{Log } H_i^{-2} \partial_j \text{Log } H_k^{-2} = 0 \quad (j \neq k.) \end{aligned} \quad (2.17)$$

THEOREM 1. *Necessary and sufficient conditions that the orthogonal coordinates $\{x^i\}$ be R -separable for the Helmholtz equation*

$$\frac{1}{h} \sum_{i=1}^n \partial_i (h H_i^{-2} \partial_i \psi) = E \psi, \quad h = H_1 \cdots H_n$$

are:

- (1) *The metric $ds^2 = \sum H_i^2 (dx^i)^2$ is in Stäckel form.*
- (2) *$\sum H_i^{-2} (f_{ii} + \frac{1}{2} f_i^2)$ is a Stäckel multiplier, where $f_i = \partial_i \text{Log}(h/S)$ and S is the determinant of the Stäckel matrix.*

If these conditions are satisfied then

$$R = (S/h)^{1/2} \prod_{i=1}^n L_i(x^i) \quad (2.18)$$

where the $L_i = L_i(x^i)$ are arbitrary.

We say that the orthogonal coordinates $\{x^i\}$ are *separable* for the Helmholtz equation provided they are R -separable with $R \equiv 1$. Furthermore, R -separable coordinates are *trivially R -separable* if $R = \prod_{i=1}^n L_i(x^i)$ and (since coordinates are trivially R -separable if and only if they are separable) we regard trivial R -separation as equivalent to ordinary separation.

Past studies of separation for (2.2) have focussed almost exclusively on

ordinary separation. For $R \equiv 1$ condition (2.12) is satisfied identically and (2.11) becomes the Robertson condition [5]:

$$\partial_{ij} \text{Log}(h/S) = 0, \quad i \neq j. \quad (2.19)$$

The Robertson condition appears to depend critically on the choice of the Stäckel matrix. However, Eisenhart [2] has shown that (2.19) is equivalent to the requirement

$$R_{ij} = 0, \quad i \neq j \quad (2.20)$$

where R_{ij} is the Ricci tensor expressed in terms of the orthogonal coordinates $\{x^k\}$.

Let us suppose that $\{x^k\}$ R -separates the Helmholtz equation. Then expanding the Stäckel matrix in (2.4) on the i th column we obtain operators A_i , $i = 1, n$, such that $A_i \psi = \lambda_i \psi$ for an R -separated solution ψ :

$$A_i = \sum_j \frac{S^{ji}}{S} \left(\partial_{jj} + f_j \partial_j + m_j + \frac{1}{2} \partial_j [f_j - l_j] + \frac{1}{4} [f_j^2 - l_j^2] \right). \quad (2.21)$$

It is convenient at this point to introduce the functions $\rho_i^{(k)}(\mathbf{x})$, where

$$S^{jk}/S = \rho_j^{(k)} H_j^{-2}, \quad 1 \leq j, k \leq n. \quad (2.22)$$

Then $\rho_j^{(1)} = 1$ and it can be shown that [1, Appendix 13; 2]

$$\partial_i \rho_j^{(k)} = (\rho_i^{(k)} - \rho_j^{(k)}) \partial_i \text{Log } H_j^{-2}. \quad (2.23)$$

Thus we have

$$A_k = \sum_j \rho_j^{(k)} H_j^{-2} (\partial_{jj} + f_j \partial_j + \xi_j), \quad 1 \leq k \leq n \quad (2.24)$$

where

$$\xi_j = m_j + \frac{1}{2} \partial_j (f_j - l_j) + \frac{1}{4} (f_j^2 - l_j^2) \quad (2.25)$$

and, using (2.11), (2.12) and (2.22), it is not difficult to verify that

$$[A_i, A_k] = 0, \quad 1 \leq i, k \leq n \quad (2.26)$$

where $[A, B] = AB - BA$. We see that the A_k , $k \geq 2$, form a commuting family of symmetry operators for \mathcal{A} , i.e., they commute with \mathcal{A} and with each other. Furthermore, the separated solutions (2.2) are simultaneous eigenfunctions of the symmetry operators.

The above construction starts with an orthogonal separable coordinate system $\{x^i\}$ and produces a commuting family of second-order symmetry

operators $\{A_k\}$. Not all families of $n-1$ commuting symmetry operators correspond to variable separation. (See [6, p. 55] for a counterexample.) In this paper we shall derive necessary and sufficient conditions on a commutative family $\{A_i\}$ in order that it correspond to an orthogonal separable coordinate system $\{x^i\}$ for (2.2) via the relations (2.24).

In Ref. [3] we solved the corresponding problem for the Hamilton–Jacobi equation (2.10). In that case one utilizes the natural symplectic structure on the cotangent bundle \tilde{V}_n of V_n . Corresponding to local coordinates $\{x^i\}$ on V_n we have coordinates $\{x^i, p_i\}$ on the $2n$ -dimensional space \tilde{V}_n . The Poisson bracket of two functions $F(x^j, p_j), G(x^j, p_j)$ on \tilde{V}_n is defined by

$$\{F, G\} = \sum_i (\partial_{p_i} F \partial_{x^i} G - \partial_{x^i} F \partial_{p_i} G). \quad (2.27)$$

Let $H = \sum_j H_j^{-2} p_j^2$ be the Hamiltonian corresponding to (2.10). If $\{x^i\}$ is an orthogonal separable coordinate system for the Hamilton–Jacobi equation then there exists a Stäckel matrix $(S_{ij}(x^i))$ such that H_j^{-2} is given by (2.9). Furthermore, the quadratic forms A_k ($A_1 = H$),

$$A_k = \sum_j \rho_j^{(k)} H_j^{-2} p_j^2, \quad k = 1, \dots, n \quad (2.28)$$

satisfy $\{A_i, A_k\} = 0$, and when evaluated for $p_j = \partial_j W$ with W a separable solution of (2.8) they satisfy $A_k = \lambda_k$, where $\lambda_1 = -E, \dots, \lambda_n$ are the separation parameters. Thus the $\{A_k\}$ form an involutive family of Killing tensors for V_n .

Let $\alpha^{ij}(\mathbf{y})$ be a symmetric contravariant 2-tensor on V_n , expressed in terms of local coordinates $\{y^k\}$, and let $g^{ij}(\mathbf{y})$ be the contravariant metric tensor. A root $\rho(\mathbf{y})$ of α^{ij} is an analytic solution of the characteristic equation

$$\det(\alpha^{ij} - \rho g^{ij}) = 0 \quad (2.29)$$

and an eigenform $\omega = \sum_k \mu_k dy^k$ corresponding to ρ is a nonzero one-form such that

$$\sum_{j=1}^n (\alpha^{ij} - \rho g^{ij}) \mu_j = 0, \quad i = 1, \dots, n. \quad (2.30)$$

Roots and eigenforms are defined independent of local coordinates [1]. If α^{ij} has n distinct roots then the corresponding eigenforms constitute an orthogonal basis for the one-forms on V_n .

Note from (2.18) that for an orthogonal separable system $\{x^j\}$ the $\rho_j^{(k)}$, $j = 1, \dots, n$, are the roots of the quadratic forms A_k and the dx^k constitute a basis of simultaneous eigenforms. In Ref. [3] we proved the following

strengthened version of a result due to Eisenhardt [1, Appendix 13]. Let $H = \sum g^{ij}(\mathbf{y}) p_i p_j$ be the Hamiltonian for (1.1).

THEOREM 2. *Necessary and sufficient conditions for the existence of an orthogonal separable coordinate system $\{x^i\}$ for the Hamilton–Jacobi equation (1.3) are that there exist n quadratic functions $A_k = \sum_{i,j} a_{(k)}^{ij} p_i p_j$ ($A_1 = H$) on \tilde{V}_n such that:*

$$(1) \quad \{A_k, A_l\} = 0, \quad 1 \leq k, l \leq n.$$

(2) *The set $\{A_k\}$ is linearly independent (as n quadratic forms).*

(3) *There is a basis $\{\omega_{(j)} : 1 \leq j \leq n\}$ of simultaneous eigenforms for the $\{A_k\}$. If conditions (1)–(3) are satisfied then there exist functions $g^j(\mathbf{x})$ such that $\omega_{(j)} = g^j dx^j$, $j = 1, \dots, n$.*

In Section 4 we will show that, with suitable modification, this result also characterizes orthogonal R -separable systems for the Helmholtz equation.

3. SELF-ADJOINT FORM

We return to the Hamiltonian operator Δ , (1.2), expressed in terms of some arbitrary local coordinate system $\{y^j\}$. Let A be a second-order symmetry operator for Δ , i.e., a differential operator such that $[A, \Delta] = 0$ and which in local coordinates can be written

$$A = \sum_{i,j} \alpha^{ij}(\mathbf{y}) \partial_{ij} + \sum_i \tilde{b}^i(\mathbf{y}) \partial_i + c(\mathbf{y}), \quad \partial_i = \partial_{y^i} \quad (3.1)$$

where $\alpha^{ij} = \alpha^{ji}$ and not all α^{ij} vanish. We can decompose A uniquely in the form

$$A = S + L \quad (3.2)$$

where

$$S = \frac{1}{\sqrt{g}} \sum_{i,j} \partial_i (\sqrt{g} \alpha^{ij} \partial_j) + c \quad (3.3)$$

$$L = \sum_i \tilde{b}^i \partial_i \quad (3.4)$$

and this decomposition is coordinate independent.

LEMMA 1. $[L, \Delta] = 0$ and $[S, \Delta] = 0$.

Proof. Let $dV = \sqrt{g} dy$ be the invariant volume element for V_n and consider the inner product

$$\langle f, h \rangle = \int_{V_n} f(y) \bar{h}(y) dV \quad (3.5)$$

where f, h are c -functions with compact support on V_n . (We allow f, h to be arbitrary except that their support must lie in the $\{y^j\}$ coordinate neighbourhood and be chosen such that the following inner products are well defined and such that the boundary terms vanish in the integration by parts formulas to follow.) Defining the adjoint A^* of A by the usual formula

$$\langle A^* f, h \rangle = \langle f, Ah \rangle, \quad \text{all } f, h \quad (3.6)$$

we verify easily that $\mathcal{A}^* = \mathcal{A}$, $\mathcal{S}^* = \mathcal{S}$, and $[A^*, \mathcal{A}] = [A^*, \mathcal{A}^*] = [\mathcal{A}, A]^* = 0$ so that A^* is a second-order symmetry operator. It follows that A can be decomposed uniquely in the form $A = A_1 + A_2$, where $[A_i, \mathcal{A}] = 0$, $i = 1, 2$, $A_1^* = A_1$, $A_2^* = -A_2$.

If $L = 0$ then $A_1 = S$ and we are done. If $L \neq 0$ choose new coordinates $\{x^j\}$ such that $L = \partial_1$. Then $A_2 = \partial_1 + \frac{1}{4} \partial_1 \text{Log } g$ and since $[A_2, \mathcal{A}] = 0$ it follows that $\partial_1 g^{ij} = 0$ for all i, j ; hence $\partial_1 \text{Log } g = 0$. We conclude that $A_2 = L$. Q.E.D.

We say that a second-order symmetry operator S is in *self-adjoint form* if $S^* = S$, that is, if S can be expressed in the form (3.3). It follows from the proof of the Lemma that every second-order symmetry A can be expressed uniquely as $A = S + L$, where S is in symmetric form and L is a first-order symmetry (automatically skew-symmetric). Finally, we say that a second-order symmetry operator S is in *reduced self-adjoint form* if S is given by expression (3.3) with c a constant. Note that if $L_j = b_{(j)}^i \partial_i$, $j = 1, 2$, are nonzero first-order symmetries then $L_1 L_2 + L_2 L_1$ is a second-order symmetry in reduced self-adjoint form.

4. THE FUNDAMENTAL RESULT

From (2.13) and (2.21)–(2.23) we see that operators A_i defining orthogonal R -separation for the Helmholtz equation are always in self-adjoint form. Thus to characterize orthogonal separation by symmetries we must necessarily restrict ourselves to operators of this form.

Let $\{y^j\}$ be a local coordinate system on V_n and let A be a second-order symmetry operator in self-adjoint form, expressed in these coordinates by

$$A = \sum \frac{1}{\sqrt{g}} \partial_i (\sqrt{g} a^{ij} \partial_j) + c. \quad (4.1)$$

Then A is uniquely associated with a quadratic form A on \tilde{V}_n and defined in local coordinates by

$$A = \sum a^{ij} p_i p_j. \quad (4.2)$$

We can talk about the *roots* and *eigenforms* of A , meaning by this the roots (2.29) and eigenforms (2.30) of A .

With these preliminaries out of the way, we present our basic result. Let A , (1.1), be the Hamiltonian operator for V_n .

THEOREM 3. *Necessary and sufficient conditions for the existence of an orthogonal R -separable coordinate system $\{x^i\}$ for the Helmholtz equation (1.1) are that there exists a linearly independent set $\{A_1 = A, A_2, \dots, A_n\}$ of second-order differential operators on V_n such that:*

$$(1) \quad [A_k, A_i] = 0, \quad 1 \leq k, i \leq n.$$

(2) *Each A_k is in self-adjoint form.*

(3) *There is a basis $\{\omega_{(j)} : 1 \leq j \leq n\}$ of simultaneous eigenforms for the $\{A_k\}$. If conditions (1)–(3) are satisfied then there exist functions $g^j(\mathbf{x})$ such that $\omega_{(j)} = g^j dx^j$, $j = 1, \dots, n$.*

Proof. Suppose conditions (1)–(3) are satisfied. Comparing coefficients of third derivative terms in condition (1) we find $\{A_k, A_i\} = 0$, where by condition (2) A_k is the quadratic form (4.2) uniquely associated with the operator A_k (4.1). It follows easily from condition (3) that the hypotheses of Theorem 2 are satisfied. Hence, there exists an orthogonal local coordinate system $\{x^i\}$, such that dx^j is a simultaneous eigenform for each operator A_k , and a Stäckel matrix $(S_{ij}(x^j))$ which defines a separation of variables for the Hamiltonian–Jacobi equation (1.3). Denoting the roots of A_k by $\rho_j^{(k)}$, $j = 1, \dots, n$, we see from the discussion in Section 2 that in the $\{x^i\}$ local coordinates

$$A_k = \sum_j \rho_j^{(k)} H_j^{-2} p_j^2 \quad (4.3)$$

where H_j^{-2} is given by (2.9) and

$$\partial_i \rho_j^{(k)} = (\rho_i^{(k)} - \rho_j^{(k)}) \partial_i \text{Log } H_j^{-2}. \quad (4.4)$$

Thus

$$A_k = \sum_j \rho_j^{(k)} H_j^{-2} (\partial_{jj} + f_j \partial_j + \xi_j) \quad (4.5)$$

where

$$f_j(\mathbf{x}) = \partial_j \text{Log} \left(\frac{H_1 \cdots H_n}{S} \right). \quad (4.6)$$

(Since the set $\{A_k\}$ is linearly independent $\det(\rho_j^{(k)}) \neq 0$ and we can express the zeroth-order term c_k of A_k in the form $c_k = \sum_j \rho_j^{(k)} H_j^{-2} \xi_j$ for some functions ξ_j .) Here

$$\sum_j H_j^{-2} \xi_j = 0 \quad (4.7)$$

since $A_l = A$.

We have not yet fully utilized condition (1): $[A_k, A_l] = 0$ for all l, k . Comparing coefficients of ∂_{ij} on both sides of this relation we obtain $\partial_i f_j = \partial_j f_i$, a requirement which is already satisfied on account of (4.6). Equating coefficients of ∂_b in condition (1) we find

$$\begin{aligned} 2\rho_b^{(k)} H_b^{-2} \partial_b \left(\sum_j \rho_j^{(l)} H_j^{-2} \xi_j \right) + \sum_j \rho_j^{(k)} H_j^{-2} (\partial_{ij} + f_j \partial_j) (\rho_b^{(l)} H_b^{-2} f_b) \\ = (k \leftrightarrow l) \end{aligned} \quad (4.8)$$

where the right-hand side of this equation is obtained from the left-hand side by interchanging the indices k and l . Utilizing (4.4) and equating coefficients of $\rho_a^{(k)} \rho_b^{(l)}$, $a \neq b$, in (4.8) we find

$$\partial_b (\xi_a - \tfrac{1}{2} f_{aa} - \tfrac{1}{4} f_a^2) = 0, \quad a \neq b$$

or

$$\xi_a = \tfrac{1}{2} (f_{aa} + \tfrac{1}{2} f_a^2) + R_a(x^a), \quad a = 1, \dots, n. \quad (4.9)$$

(The equating of the zeroth-order terms in condition (1) yields a relation which is already satisfied.) Substituting (4.9) into (4.7) we see that $\sum_a H_a^{-2} (f_{aa} + \tfrac{1}{2} f_a^2)$ is a Stäckel multiplier. Thus the conditions of Theorem 1 are satisfied and the local coordinates $\{x^i\}$ R -separate the Helmholtz equation.

Conversely, if the orthogonal coordinates $\{x^i\}$ R -separate the Helmholtz equation it is easy to show, using (2.21) and (4.9), that conditions (1)–(3) are satisfied. Q.E.D.

5. EXAMPLES OF R -SEPARATION

The phenomenon of R -separation for the Helmholtz equation has received very little notice. Indeed the only previous reference we have located to date is [7], in which Moon and Spencer define orthogonal R -separation for Euclidean space. However, since for Euclidean space (even all Einstein spaces) the Robertson condition $R_{ij} = 0$, $i \neq j$, is automatically satisfied it

follows from (2.18), (2.19) that [8]: An orthogonal R -separable system in an Einstein space is trivially R -separable. The fact that nontrivial orthogonal R -separation does not occur in flat space and spaces of constant curvature undoubtedly accounts for the lack of notice given to this phenomenon.

To our knowledge the first published example is contained in the note [9]. The space is three dimensional and conformally flat. The metric in the R -separable coordinates is

$$ds^2 = (x + y + z)[(x - y)(x - z) dx^2 + (y - z)(y - x) dy^2 + (z - x)(z - y) dz^2]. \quad (5.1)$$

The multiplier is

$$R = (x + y + z)^{-1/4}. \quad (5.2)$$

An even simpler example, however, is provided by the coordinates $\{x, y, z\}$ on the space with metric

$$ds^2 = -dx^2 + dy^2 + (y - x)^{-1} dz^2. \quad (5.3)$$

Here

$$R = (x - y)^{1/4}. \quad (5.4)$$

Both of these examples have the feature that the corresponding symmetry operators contain nonvanishing zeroth-order terms, i.e., the operators are not reduced. It is of interest to determine if nontrivial R -separation is possible in which the symmetry operators are reduced. We see from (2.21) that necessary and sufficient conditions for the symmetry operators to be obtainable in reduced form are

$$\partial_{ij} f_i + f_i \partial_j f_i = 0, \quad i, j = 1, \dots, n, i \neq j \quad (5.5)$$

where

$$f_i = \partial_i \log(H_1 \cdots H_n / S). \quad (5.6)$$

Indeed, this follows directly from

LEMMA 2. *Let $(S_{ij}(x^i))$ be a Stäckel matrix with cofactors S^{ij} and determinant S , respectively, and such that each $S^{i1} \neq 0$. Further, let $\xi_i(\mathbf{x})$ be an n -tuple of functions such that $\sum_i S^{i1} \xi_i(\mathbf{x}) = 0$. Then $\sum_i (S^{ij}/S) \xi_i(\mathbf{x})$ is a constant for each $j = 2, \dots, n$ if and only if $\xi_i = \xi_i(x^i)$ for each $i = 1, \dots, n$.*

Proof. Since $\sum_i S^{il} \xi_i(\mathbf{x}) = 0$ there must exist functions $C_j(\mathbf{x})$ such that

$$\xi_i(\mathbf{x}) = \sum_{j=2}^n C_j(\mathbf{x}) S_{ij}(x^i). \quad (5.7)$$

Now

$$\sum_i \frac{S^{ik}}{S} \xi_i(\mathbf{x}) = \sum_{i,j} \frac{S^{ik}}{S} S_{ij}(x^i) C_j(\mathbf{x}) = C_k(\mathbf{x})$$

so if $C_k(\mathbf{x})$ is always constant we have $\xi_i = \xi_i(x^i)$. On the other hand, if $\xi_i = \xi_i(x^i)$ it follows from (5.7) that

$$0 = \sum_{j=2}^n \partial_l C_j(x) S_{ij}(x^i), \quad i \neq l.$$

Since $S^{il} \neq 0$ we have $\partial_l C_j = 0$. Thus C_j is a constant.

Q.E.D.

System (5.5) can be readily solved. If we assume $\partial_j f_i \neq 0$ for all $i \neq j$ then these equations can be integrated to give

$$\partial_{ij} f = A e^{-f}$$

where $\partial A / \partial x^l = 0$, $l = i, j$. If we write $f = -g + \log A + i\pi$ then g satisfies Liouville's equation [10]

$$\partial_{ij} g = e^g$$

which has the general solution

$$e^g = 2 \frac{\alpha'_i(x^i) \alpha'_j(x^j)}{(\alpha_i(x^i) + \alpha_j(x^j))}. \quad (5.8)$$

For this to hold for all combinations of i, j (with $\partial_i f_j \neq 0$) we must have

$$e^f = \frac{(\sum_{i=1}^n \alpha_i(x^i))^2}{\prod_{i=1}^n \alpha'_i(x^i)}. \quad (5.9)$$

This is the multidimensional analogue of the relationship between the equation $\partial_{12} h' = 0$ and $\partial_{12} h' = e^{h'}$ (Liouville's equation). The system of equations $\partial_{ij} h = 0$, $\partial_i h \neq 0$, $i, j = 1, \dots, n$, with general solution $h = \sum_{i=1}^n \alpha_i(x^i)$ gives rise to the solution (5.9) of the nonlinear system (5.5). For the general solution of (5.5), when $\partial_i f_j$ may vanish for some choices of

i, j , we must divide the set $\{1, \dots, n\}$ into mutually disjoint subsets η_α , $\dim \eta_\alpha \geq 2$ ($\alpha = 1, \dots, N$) such that

$$\bigcup_{\alpha=1}^N \eta_\alpha = \{1, \dots, n\}. \quad (5.10)$$

The general solution of the system (5.5) is

$$e^f = \prod_{i \in \eta_1} h_i(x^i) \prod_{\alpha=2}^N \left\{ \frac{(\sum_{j \in \eta_\alpha} \alpha_j(x^j))^2}{\prod_{j \in \eta_\alpha} \alpha'_j(x^j)} \right\} \quad (5.11)$$

where $e^f = (\prod_{l=1}^n S^{l1})/S^{n-2}$ and we have written the metric in Stäckel form. In fact by suitably redefining variables x^i we can always choose restriction (5.5) as

$$\frac{\prod_{l=1}^n S^{l1}}{S^{n-2}} = \prod_{\alpha=2}^N \left(\sum_{j \in \eta_\alpha} x^j \right)^4. \quad (5.12)$$

This expression makes it particularly straightforward to find a metric which satisfies (5.5). One such example is the metric

$$ds^2 = (x^1 + x^2) \left[\sum_{i=1}^6 (dx^i)^2 \right]. \quad (5.13)$$

The Helmholtz equation becomes

$$\Delta \psi \equiv \frac{1}{x^1 + x^2} \left[\partial_{11} + \partial_{22} + \frac{2}{x^1 + x^2} (\partial_1 + \partial_2) + \sum_{l=3}^6 \partial_{ll} \right] \psi = E \psi. \quad (5.14)$$

If we write $\psi = (x^1 + x^2)^{-1} \emptyset$ then \emptyset satisfies

$$\sum_{l=1}^6 \partial_{ll} \emptyset = (x^1 + x^2) E \emptyset \quad (5.15)$$

which is clearly separable in these coordinates. The symmetry operators describing the R -separation are A_l , $l = 3, \dots, 6$, and B where

$$\begin{aligned} A_l &= \partial_{ll}, \\ B &= \frac{1}{x^1 + x^2} \left[x^2 \left(\partial_{11} + \frac{2}{x^1 + x^2} \partial_1 \right) - x^1 \left(\partial_{22} + \frac{2}{x^1 + x^2} \partial_2 \right) \right. \\ &\quad \left. + \frac{1}{2} (x^2 - x^1) \sum_{l=3}^6 \partial_{ll} \right] \end{aligned} \quad (5.16)$$

and these operators are in reduced self-adjoint form.

6. DISCUSSION OF RESULTS

The results of this paper lead to several important conclusions concerning (orthogonal) variable separation and R -separation for the Helmholtz equation. First, one must recognize the intrinsic geometrical nature of R -separation. From Theorem 1 it appears that the conditions for R -separation are highly technical and nongeometric. However, Theorem 3 shows that these conditions are equivalent to the existence of an n -dimensional family of commuting self-adjoint symmetry operators which can be simultaneously diagonalized.

Second, comparing Theorem 2 and 3, it is obvious that orthogonal R -separation, not ordinary separation, for the Helmholtz equation is the natural analogy of additive orthogonal separation for the Hamilton–Jacobi equation. Furthermore, with the more powerful tool of R -separation we expect to find new useful solutions of Schrödinger equations with potentials.

Finally, a comparison of Theorems 2 and 3 shows the close relationship between variable separation and quantization theory. Corresponding to a separable system $\langle x^j \rangle$ for the Hamilton–Jacobi equation we have an involutive family $\{A_I\}$ of quadratic constants of the motion

$$A_I = \sum a_{(I)}^{ij} p_i p_j.$$

The Helmholtz equation R -separates in these same coordinates if and only if functions c_I can be found such that the operators

$$A_I = \frac{1}{\sqrt{g}} \sum \frac{\partial_i (\sqrt{g} a_{(I)}^{ij} \partial_j) + c_I}{ij}$$

pairwise commute. The requirement that $\sum H_i^{-2} (f_{ii} + \frac{1}{2} f_i^2)$ is a Stäckel multiplier is the precise condition that this construction can be carried out.

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